Quantum Black Holes:Unexpected Results

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Abstract

The quantum black hole model with a self-gravitating spherically symmetric thin dust shell as a source is considered. The shell Hamiltonian constraint is written and the corresponding Schroedinger equation is obtained. This equation appeared to be a finite differences equation. Its solutions are required to be analytic functions on the relevant Riemannian surface. The method of finding discrete spectra is suggested based on the analytic properties of the solutions. The large black hole approximation is considered and the discrete spectra for bound states of quantum black holes and wormholes are found. They depend on two quantum numbers and are, in fact, quasi-continuous. The quantum black hole bound state depends not only on mass but also on an additional quantum number, and black holes with the same mass have different quantum hairs. These hairs exhibit themselves at the Planckian distances near the black hole horizon. For the observer who can not measure the distances smaller than the Planckian length the black hole has the only parameter, its mass. The other, non-measurable, parameter leads to the quantum corrections to the black hole entropy. The quantum states with given mass are the mixed ones. It is shown that there exists the ground quantum black hole state with minimal mass equal approximately the Planckian mass. Its quantum state has zero entropy and it is a pure state. The existence of the quantum hairs may solve (at least partially) the well known information paradox in the black hole physics.

[1]

[1]Talk presented at the Birthday Conference dedicated to A.Arvilski, 17th of February, 1998

1. Classical theory. We start with description of our model. It is a self-gravitating spherically symmetric thin dust shell endowed with bare mass M. Note that the shell is not embedded into the Schwarzschild manifold in which case it can be considered as some set of test particles (observers). Our shell serves as a source of a gravitational field. Inside the shell the space-time is Minkowskian, and outside it is Schwarzschildean with mass m. In what follows we need some well known facts from the classical theory of black holes. Every spherically symmetric space-time can be locally characterized by two invariant functions of two variables (some time coordinate t and some radial coordinate r). These are the radius of a sphere $R(t, r)$ and the differential invariant $F(t, r) = g^{\alpha,\beta} R_{,\alpha} R_{,\beta}$, the latter being equal to $F = 1 - 2Gm/R$ in the Schwarzschild case. The complete Schwarzschild manifold consists of four parts characterized by the signs of function F and the signs of partial derivatives of $R(t, r)$, they are called $R_\pm$- and $T_\pm$-regions. In the R-regions $F > 0$, and $R' > 0$ in R_+-region (R ranges between the event horizon $R_g = 2Gm$ and infinity) and $R' < 0$ in R_--region ($\infty > R > R_g$). In the T-regions $F < 0$, and the T_+-region in which $\dot{R} > 0$ is called the region of inevitable expansion while the T_--region with $\dot{R} < 0$ is called the region of inevitable contraction. We are interested here in the bound motion only. So, a trajectory of our shell has a turning point of radius R_0 which can be located in one of the R-regions (but not in the T-regions). It appears that if the ratio of the total (Schwarzschild) mass m and the bare mass M is in the range $1/2 < m/M < 1$, then the turning point lies in the R_+-region, we call this a black hole case. If $m/M < 1/2$, then the turning point is in the R_--region and we call this a wormhole case. The "black hole" shell does not enter the R_--region, and the "wormhole" shell does not enter the R_+-region. The main feature of the "black hole shells is that for fixed R_0 the larger the bare mass, the larger the total mass, i.e. $\frac{\partial m}{\partial M} > 0$. For the "wormhole" shells $\frac{\partial m}{\partial M} < 0$. It is interesting to note that in the R_--region outside the wormhole there can exist (even classically) the shell with the negative total mass (that is, $m_{out} < m_{in}$). We can also insert in the R_--region two shells with negative and positive masses equal up to the sign. or the shell with negative mass may be placed in the R_--region while the shell with positive mass may lie in the R_+-region [1],[2]. In quantum theory such shells could be created spontaneously causing the Hawking radiation and instability (the so called Klein paradox).

2. Hamiltonian picture. To construct a quantum theory of black holes and wormholes we need a classical geometrodynamical description of our

model. The geometrodynamics of the eternal Schwarzschild black hole (both classical and quantum) was considered in full details by K.Kuchar [3]. The geometrodynamics of the general spherically symmetric space-time with the thin shell as a source was constructed in our paper [4]. It was shown that the corresponding Hamiltonian constraint for the shell depends only on the invariant functions R and F, on the bare mass of the shell M and the momentum P_R conjugate to the variable R. It can be written in the form

$$C = F + 1 - \sqrt{F}\left(\exp\frac{GP_R}{R} + \exp -\frac{GP_R}{R}\right) - \frac{M^2G^2}{R^2} = 0 \qquad (1)$$

Strictly speaking, the above equation was derived for R_+-region only. Because $\sqrt{F}$ it is not valid as it is in T-regions. Of course, the analogous equations can also be derived separately for T-regions. But, having in mind that in quantum theory it is desirable to have a single wave function for all the four patches of the complete Schwarzschild manifold we have chosen quite a different way. We consider $f = \sqrt{F}$ as a function complex variable, namely, $f = |F|^{1/2}e^{i\phi}$, which has branching points at the horizons, when $F = 0$. We choose the following rules of bypassing around these branching points. In the black hole case $\phi = 0$ in the R_+-region, $\phi = \frac{\pi}{2}$ in the T_--region, $\phi = \pi$ in the R_--region and $\phi = -\frac{\pi}{2}$ in the T_+-region. In the wormhole case the bypass goes in the opposite direction starting from the R_--region. The Hamiltonian constraint is now a complex valued function but it can easily be made real by adding the relevant complex conjugate part. Some words are in order here. In classical theory we can write different Hamiltonian constraints which are more simple than the above one. The examples can be found in [5] (the quantum equation is the equation in finite differences but an exactly solvable one) and in [6] (the Hamiltonian is quadratic in momenta, and its quantum counterpart is a Klein-Gordon equation). All these and other Hamiltonians lead to the same classical motion for the shell, and the space-time geometry can be (locally) reconstructed (see [2] for rather general discussion). But in the quantum theory there are no trajectories and such a reconstruction of the space-time geometry is impossible. Thus, the geometric structure of an underlying manifold should be incorporated into the structure of the quantum Hamiltonian constraint itself. The use of the more simple constraint than the Eqn.(1) leads to the identification of the R_-- and R_+-regions and of the T_+- and T_--regions. But the quantum motion (unlike the classical one) is possible both in R_+- and R_--regions for the same values of the mass parameter. Thus, such an identification leads to the essential loss of the information about the space-time geometry. The

advantage of our analytical continuation is not only that we can now obtain a single wave function for a quantum self-gravitating shell, but what is more important the R_+- and R_--regions of the complete Schwarzschild manifold are no more identical but can be considered as lying on different leaves of Riemannian surface of complex variable $f(=|F|^{1/2}e^{i\phi})$. We will see soon that this fact affects the quantum mass spectrum very much. We are not going to consider here the classical evolution which comes from the above Hamiltonian, but will jump directly to the quantum picture.

3. Quantization. In the quantum theory both the variables and their conjugate momenta become operators, and the Hamiltonian constraint acts on the wave function as operator. In our case it is more convenient to use the radius R and its conjugated momentum P_R but the equivalent canonical pair $s = R^2/R_g^2$ and $P_s = \frac{P_g^2}{2R} P_R$. In the coordinate representation the wave function Ψ depends only on s, which is a multiplication operator , and P_s becomes a differential operator $P_s = -i\frac{\partial}{\partial s}$. The exponential operator $exp(GP_R/R) = exp(-i\xi\frac{\partial}{\partial s})$ which enters our Hamiltonian constraint becomes in the coordinate representation an operator of finite shift.

$$e^{-i\xi\frac{\partial}{\partial s}}\Psi = \Psi(s - \xi i) \tag{2}$$

where m_{pl} is Plank mass and $\xi = \frac{1}{2}(\frac{m_{pl}}{M})^2$. Thus , we arrive at the following quantum equation for a self gravitating thin dust shell

$$\begin{gathered} f\left(\Psi(s+i\xi)+\Psi(s-i\xi)\right)+\overline{f}(s+i\xi)\Psi(s+i\xi)+ \\ \overline{f}(s-i\xi)\Psi(s-i\xi) = 2(F+1-\frac{m^2}{4m_{Pl}^2 s})\Psi(s) \end{gathered} \tag{3}$$

where $f = |F|^1/2e^{i\phi}$ and we have chosen a symmetric operator ordering with an appropriate complex conjugation. The quantum equation obtained is the equation in finite differences rather than the differential one as in ordinary quantum mechanics, and the shift is along imaginary axis. The quantum mechanical postulates tell us that the Hamiltonian (as the constraints as well) should be self-adjoint operators. And this goal is achieved usually by imposing appropriate boundary conditions on the wave functions. As by product, for bound states we obtain usually a discrete energy (mass) spectrum. The reason for this is that for any homogeneous ordinary differential equation , say, of second order we need only one condition to single out the solution (up to the renormalization factor). But for the corresponding quantum operator to be a self-adjoint we need two boundary conditions for bound states . It is such an extra condition which leads to the discrete

spectrum . The situation is different in the case of our equation in finite differences . In the paper [5] the toy quantum black hole model was considered. The quantum equation in this model is also equation in finite differences. This equation appeared to be exactly solvable and it was shown that it was possible to obtain a self-adjoint extension of the of the corresponding Hamiltonian by imposing of the countable number of boundary conditions on the wave functions. All these boundary conditions allowed to single out the solution (up to the inherent degeneracy), but they did not allow to obtain the discrete spectrum. We expect our problem to have a discrete mass (energy) spectrum for bound states because the same problem in the nonrelativistic limit has this feature. How to obtain it ? Fortunately , we have one more requirement the wave function should satisfy. The solution to the second order differential equation should be differentiable twice (at least). But now we have at hand an equation in finite differences. Moreover the shift is along the imaginary axis. Therefore , we must require that the solution should be analytical function. The analyticity is a very strong feature. Our equation has singular points, in particular, the points at the horizons (s=1) are the branching points. The solutions , in general , will have branching points too. And the types of these branching points do not depend neither on the particular operator ordering nor on the boundary conditions imposed on the wave functions to ensure the self-adjointness of th Hamiltonian (but the wave function should still be integrable with square). Moreover,in order to be able to construct the wave function which is single-valued on some Riemannian surface, we should have the branching points of the same type (when this points can be connected by the cuts). It is the comparison of the branching points of the "good" (integrable) solutions that will lead us to the discrete spectra for bound states. We will see in a moment how such a procedure works in the limit of large black holes (e.i., large total mass m.).

4. Large black holes We would like to illustrate the ideas described above by considering the limiting case of small values of ξ. Since $\xi = \frac{1}{2}(m_{pl}/m)^2$, it is not only the limit of black holes with masses much larger than the Planckian mass, but at the same time it is a quasiclassical limit because m_{pl}^2 is proportional to the Planckian constant $\hbar$. In this limiting case we can expand any function of shifted argument in the following series

$$\Phi(s+\xi i) = \Phi(s) + i\xi\Phi'(s) - \frac{\xi^2}{2}\Phi''(s) + ... \tag{4}$$

We cut these series at the second order in ξ. Here we present only the results of our investigation. The resulting equations are different for different parts

of the Schwarzschild manifold but all of them have the same singular points as the original equation. These points are $s \to \infty$, $s \to 1+0$ in $R_\pm$-regions, and $s \to 1-0$ in $T_\pm$-regions (our expansion is not valid near $s=0$ and this point is not relevant to the results). As was explained before we need only to know the asymptotic behavior of the solutions near singular points of the equations. Below we consider the black hole case only. The results are readily translated to the wormhole case. The very interesting feature of our equation is the fact that the approximate differential equations in $R_\pm$-regions are of the second order, while in $T_\pm$-regions the equations appear to be of the first order ones. The corresponding asymptotics at $s \to 1-0$ are

$$\Psi \sim \exp\left(i \frac{8}{3\xi^2} \left(1 - \frac{M^2}{4m^2} \right) (-y)^{3/2} \right) \tag{5}$$

in the T_--region, and

$$\Psi \sim \exp\left(-i \frac{8}{3\xi^2} \left(1 - \frac{M^2}{4m^2} \right) (-y)^{3/2} \right) \tag{6}$$

in the T_+-region. The variables s and y are related by $s = (1+y)^2$. Thus, y is the deviation from the horizon. We see that the wave function is the radial part of the ingoing wave in the T_--region and it is that of the outgoing wave in the T_+-region. This is a quasiclassical reflection of the fact that classically the shell can only expand in the T_+-region and shrink in the T_--region. Moreover it shows that our choice of the quantum Hamiltonian constraint (operator ordering, complex conjugation etc.) gives a good quasi-classics. Let us remind that the solution to the original equation in finite differences should be an analytical function. The solution to the approximate equation should not have, of course, this feature. But the asymptotic solution on one side of the branching point should be the analytical continuation of the solution on the other side. This dictates the choice of one of the two asymptotics in the R_+-region

$$\Psi \sim 1 - \frac{8}{3\xi^2} \left(1 - \frac{M^2}{4m^2} \right) y^{3/2} \tag{7}$$
$$y \gg \xi, \quad \xi \ll 1$$

and in the R_--region

$$\Psi \sim 1 - \frac{8}{3\xi^2}\left(1 + \frac{M^2}{4m^2}\right) y^{3/2} \tag{8}$$
$$y \gg \xi, \quad \xi \ll 1$$

And, at last, the asymptotics at $s \to \infty$ in the R_+-region is

$$\begin{aligned} \Psi &\sim y^{\frac{1}{2} - \frac{\frac{M^2}{m^2} - 2}{4\mu\xi^2}} \exp(-\mu y), \\ \mu &= \frac{1}{\xi}\sqrt{\frac{M^2}{m^2} - 1}, \; y \gg \xi \end{aligned} \tag{9}$$

while in the R_--region it is

$$\Psi \sim y^{\frac{\frac{M^2}{m^2} - 1}{8\xi}} \exp(-\frac{2}{\xi} y^2) \tag{10}$$

Note that the falloff in the R_--region is much more fast than in the R_+-region.

5. Discrete mass spectrum The last step on the way to the discrete mass spectrum for bound states of black holes and wormholes is to compare different branching points of solutions, namely, at infinities and near the horizons. But before doing this we would like to discuss some new and important point. Classically, given some total mass m, we have two types of motion depending on the value of mare mass M. In the first, black hole, case the bound motion starts from the past singularity $R = 0$ in the T_+-region, has its turning point in the R_+-region and then goes to the future singularity $R = 0$ in the T_--region. The bare mass is such that $1 > \frac{m}{M} > \frac{1}{2}$, and $\frac{\partial m}{\partial M} > 0$. In the second, wormhole, case the turning point lies in the R_--region with $\frac{m}{M} < \frac{1}{2}$, and $\frac{\partial m}{\partial M} < 0$. Quantum theory changes the situation radically. As we have seen the wave function in the black hole case is not zero not only in the R_+-region, but also in the R_--region, though with relatively negligible amplitude. It means that the black hole type shell, starting from the T_+-region, may go not only through the "true" R_+-region, but also through the "wrong" R_--region (the same is true, of course, for

the wormhole type with interchange of "true" and "wrong" regions). Since, by our construction, R_+-region and R_--region lie on different leaves of the Riemannian surface, it means that the quantum shells have two degrees of freedom (contrast to the one degree for the classical shells). Consequently, the discrete mass spectrum should depend on two quantum numbers. And, indeed, the comparison of branching points of the solutions at $s \to \infty$ and $s \to 1+0$ in the R_+-region gives us the first quantum number,

$$\frac{2 - \dfrac{M^2}{m^2}}{4\zeta\sqrt{\dfrac{M^2}{m^2} - 1}} = n, \quad n = integer \tag{11}$$

Doing the same in the R_--region we obtain the second quantum number,

$$\frac{\dfrac{M^2}{m^2} - 1}{8\zeta} = \frac{1}{2} + p, \qquad p = positive \quad integer \tag{12}$$

It should be noted that the same result can be obtained by usual quasiclassical methods, so it is valid approximately for small black hole also. The quantum number n is a quantum counterpart of the classical turning point, so for fixed n it should be $\frac{\partial m}{\partial M} > 0$ in the black hole case and $\frac{\partial m}{\partial M} < 0$ in the wormhole case. It can be shown that positive values of n corresponding to the "black hole" shell with $1 < \frac{M^2}{m^2} < 2$ (instead of 4 for classical shells), while negative values of n correspond to the "wormhole" shell $\frac{\partial m}{\partial M} > 4.2$ (instead of 4). Let us denote by q the ratio $q = (1+2p)/n$. Then in the black hole case ($n \geq 0$) for $0 < q \ll 1$ we have

$$m \approx \sqrt{2}(1+2p)^{1/6} n^{1/3}\ m_{pl} \tag{13}$$

This corresponds to the shell which classical turning point is far away from the horizon. In the opposite case when the classical turning point is near the horizon we have $q \gg 1$ and

$$m \approx \sqrt{2}\sqrt{1+2p}\ m_{pl} \tag{14}$$

For $n = 0$ this approximate expression becomes exact and for $p = 0$ we obtain the minimal possible mass for black holes

$$m = \sqrt{2}\ m_{pl} \tag{15}$$

In the wormhole case the minimal value of q is $q = -3\sqrt{3}/2$, then $M^2/m^2 \approx 4$, the classical turning point is near the horizon and

$$m \approx \frac{2}{3^{1/4}} \sqrt{|n|}\, m_{pl} \tag{16}$$

In the opposite case, $|q| \gg 1$ we have

$$m \approx \sqrt{2} \frac{|n|}{(1+2p)^{1/2}}\, m_{pl} \tag{17}$$

We can consider this as a spectrum of the nearly closed worlds.

6. Entropy and quantum hairs. The appearance of the second quantum number means that the mass spectrum of quantum black holes is, in fact, quasi-continuous. It means also that the mass is not the only parameter that describes quantum black hole states. There exists quantum hairs, different for different black holes of the same mass. But these hairs exist on the Planckian distances from the black hole horizon. Let us assume that some observer can not measure distances smaller that the Planckian length ("natural coarse graining"). Then, for such an observer, a black hole is characterized by only one parameter, its mass. Thus, the quantum state with given mass is a mixed state with nonzero entropy. The black hole with minimal positive mass has, of course, zero entropy. This entropy can be roughly estimated as follows. Consider a black hole with some (allowed) mass m_0. The lowest possible value of quantum number n equals zero for black holes/ Then,

$$m = \sqrt{2}\sqrt{1+2p}\, m_{pl} \tag{18}$$

where p_0 is the maximal possible value of the second quantum number for p_0 . Thus, the number of possible states is $N \approx p_0$. For the entropy we have

$$S \approx lnp_0 \sim 2lnm_0 \sim A \tag{19}$$

where A is a dimensionless area of the black hole horizon. In our model we are dealing with only one shell. In a more realistic model we can have many shells, its number is restricted, of course, by the mass of the black hole. Hence, we will have some number of possible black hole states which comes from the combinatorics of all possible shells times the number of specific states originated from the existence of our second quantum number. The combinatorics leads, as it is always assumed, to the well known classical

value of the black hole entropy. Thus, the full entropy, including quantum corrections, is of the form

$$S = \frac{1}{4}A + \alpha lnA \tag{20}$$

From the first law of black hole thermodynamics we obtain the following modification of the famous Hawking temperature

$$\Theta = \frac{1}{8\pi\frac{m_0}{m_{Pl}^2} + \frac{2\alpha}{m_0}} \tag{21}$$

Unlike the Hawking temperature our expression reaches the maximum. This maximum can be expected to lie near minimal black hole mass. This means that a black hole with nearly minimal mass has a positive heat capacity. Thus, small black holes are stable against the catastrophic growth due to absorption of the thermal energy. Such a feature may have important cosmological consequences. And, at the end, some more words about quantum black hole hairs. Let us now assume that some very clever observer does able to measure these quantum hairs. Then, the fine structure of the black hole mass spectrum enables us to investigate the inner structure of the black holes. It seems quite possible that this may solve the well known information paradox in the black hole physics. Moreover we are able in this case to measure the inner structure of a wormhole and extract its energy up to the very end, that is to the completely closed world with zero mass.

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